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Quantum dynamical aspects of two-dimensional spaces of constant negative curvature

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Abstract. We study the quantum mechanics of two-dimensional spaces of constant negative curvature. The connection between different formulations is analysed. We derive the propagator and find that in the semiclassical approximation, the time evolution of observables exhibits similar features as in the classical case.

1. Introduction

In modern theories of particle physics, such as superstrings or supergravity, chiral matter fields invariably parametrise a curved Kähler manifold. Apart from perturbative treatments [1], little is known about the quantum behaviour of such field theories with non-minimal kinetic terms. In the one-dimensional case such systems correspond to the motion of particles in curved spaces. Quantum mechanics on curved spaces is still an open field with many unanswered questions.

A typical example, and perhaps the simplest, is the case of an SU(1, 1)/U(1) space parametrised with a simple complex field. It is related to the dilaton and is contained in all $N=1$ supergravity theories that appear in the local limit of compactified ten-dimensional or four-dimensional superstrings. The one-dimensional case of this field theory corresponds to a point particle moving in a two-dimensional space of constant negative curvature. In this paper we study the spectrum and compute the propagator that determines the evolution of amplitudes of such a system. In previous papers [2], where we studied the classical system and some of its cosmological implications, we noted the characteristic divergence of geodesics, a feature that is well known to be related to chaoticity. In the quantum case similar features emerge when we study the time evolution of expectation values on the full non-compact manifold.

2. General considerations

Consider an N-dimensional manifold viewed as a set of points in patchwise one-to-one correspondence with open subsets of the Euclidean space R^N. Let us denote the coordinates as $\phi^1, \ldots, \phi^N$. We assume the existence of a smooth symmetric positive definite metric $g_{ij}(\phi)$ in terms of which we define a line element, a connection and a curvature tensor as

$$\mathrm{d}s^2 = g_{ij}(\phi)\,\mathrm{d}\phi^i\,\mathrm{d}\phi^j$$

$$\Gamma^i_{jk} = \tfrac{1}{2}g^{im}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk})$$

$$R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{lm}\Gamma^m_{kj}.$$

In the case $N = 2n$, a set of n complex coordinates can be introduced as $Z^A = \phi^A + \mathrm{i}\phi^{A+n}$ and $\bar{Z}_A = \bar{Z}^{\bar{A}} \equiv \phi^A - \mathrm{i}\phi^{A+n}$ $(A = 1, \ldots, n)$. In a so-called Hermitian manifold there exists a preferred class of coordinate systems for which $g_{AB} = g_{\bar{A}\bar{B}} = 0$ and $\mathrm{d}s^2 = 2g_{A\bar{B}}\,\mathrm{d}Z^A\,\mathrm{d}\bar{Z}^{\bar{B}} = 2g^B_A\,\mathrm{d}Z^A\,\mathrm{d}\bar{Z}_B$. In the special case in which

$$g_{A\bar{B}} = \frac{\partial^2}{\partial Z^A \partial \bar{Z}_B} G(Z, \bar{Z})$$

the manifold is called a Kähler manifold and the real function $G(Z, \bar{Z})$ is called a Kähler potential. Chiral superfields coupled to $N = 1$ supergravity automatically span a Kähler manifold. For a Kähler manifold the only non-vanishing components of the connection are the unmixed ones Γ^A_{BC} and $\Gamma^{\bar{A}}_{\bar{B}\bar{C}}$. In addition we can show that

$$\Gamma^B_{AB} = \partial_A \ln(\det g_{r\bar{\Delta}})$$

which leads to

$$R_{A\bar{B}} = -\partial_A \partial_{\bar{B}} \ln(\det g_{r\bar{\Delta}})$$

for the Ricci tensor $R_{A\bar{B}} = R^C_{AC\bar{B}}$. As an example of a Kähler manifold, consider the homogeneous space $\mathrm{SU}(n, 1)/\mathrm{SU}(n) \times \mathrm{U}(1)$ with a Lagrangian

$$\mathscr{L} = \frac{1}{1 - K^2 \bar{Z}_A Z^A}\left(\delta^A_B + K^2 \frac{Z^A \bar{Z}_B}{1 - K^2 Z^A \bar{Z}_A}\right)\partial_\mu Z^B \partial^\mu \bar{Z}_A \tag{2.1}$$

defined in terms of the Kähler potential $G = -\ln(1 - k^2 Z^A \bar{Z}_A)/k^2$. It is easy to check that this is a space of constant negative curvature (see appendix 1)

$$R^B_A = -(n-1)k^2 g^B_A. \tag{2.2}$$

Consider now the simplest possible case of a point particle moving freely in a general Riemannian manifold with a line element $\mathrm{d}s^2 = g_{ij}\,\mathrm{d}q^i\,\mathrm{d}q^j$. From the Lagrangian

$$L = \tfrac{1}{2} g_{ij} \dot{q}^i \dot{q}^j \tag{2.3}$$

we obtain the equations of motion

$$\ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = 0. \tag{2.4}$$

The distance travelled by the particle along a geodesic in a time interval $[0, T]$ is obtained by substituting the solution of the equations of motion in

$$D[q^i(T), q^i(0)] = \int_0^T \mathrm{d}t\,(g_{ij}\dot{q}^i\dot{q}^j)^{1/2}. \tag{2.5}$$

Since the energy is a constant of the motion $D = T\sqrt{2E}$, and therefore the value of the classical action is connected to the geodesic distance through the formula

$$S_c[q^i(T), q^i(0)] = \int_0^T \mathrm{d}t\,\tfrac{1}{2} g_{ij}\dot{q}^i\dot{q}^j = ET = D^2/2T. \tag{2.6}$$

A Hamiltonian is defined in terms of the canonical momenta $\pi_i = g_{ij}\dot{q}^j$ as $H = \frac{1}{2}g^{ij}\pi_i\pi_j$. To quantise such a system, we can either impose canonical commutation relations $[q^i, \pi_j] = \mathrm{i}\hbar\delta^i_j$ or consider the Feynman path integral. In any case the definition of the Hilbert space must be made in terms of the inner product

$$\langle\psi|\chi\rangle = \int \mathrm{d}^k q\, g^{1/2}\psi^*(q)\chi(q) \tag{2.7}$$

where $g = \det(g_{ij})$ appears in the invariant volume element $\mathrm{d}^n q\sqrt{g}$. In the $\{q^i\}$ representation the momenta are represented by the operators

$$\pi_i = -i\hbar g^{-1/4}\frac{\partial}{\partial q_i}g^{1/4}. \tag{2.8}$$

Hermiticity is proven as follows:

$$\begin{aligned}\langle\psi|\pi_i|\chi\rangle &= -\mathrm{i}\hbar\int \mathrm{d}^k q\, g^{1/2}\psi^* g^{-1/4}\frac{\partial}{\partial q^i}(g^{1/4}\chi)\\ &= -\mathrm{i}\hbar(\psi^x g^{1/2}\chi)_\infty + \mathrm{i}\hbar\int \mathrm{d}^k q\, g^{1/2}\chi g^{-1/4}\frac{\partial}{\partial q^i}(g^{1/4}\psi^*)\\ &= \langle\chi|\pi_i|\psi\rangle^*\end{aligned}$$

provided $\psi^* g^{1/2}X$ vanishes at infinity.

In the transition from the classical to the quantum Hamiltonian care should be taken of the operator ordering. With the usual midpoint definition of the Feynman path integral the various terms in the classical action correspond to Weyl-ordered operators on the Hilbert space. The relation between the differently ordered quantum Hamiltonians is not always simple. For example, the Laplacian operator, in terms of our representation of momenta, is

$$\begin{aligned}-\frac{\hbar^2}{2}&g^{-1/2}\frac{\partial}{\partial q^i}g^{ij}g^{1/2}\frac{\partial}{\partial q^j}\\ &= \tfrac{1}{2}g^{-1/4}\pi_i g^{ij}g^{1/2}\pi_j g^{-1/4}\\ &= \tfrac{1}{2}(\pi_i\pi_j g^{ij})_{\mathrm{W}} + \frac{\hbar}{4}\left[\frac{\partial}{\partial q^i}\frac{\partial}{\partial q^j}g^{ij} - 2g^{-1/4}\frac{\partial}{\partial qj}\left(g^{ij}g^{1/2}\frac{\partial}{\partial qi}g^{-1/4}\right)\right].\end{aligned}$$

For the simple Lagrangian $L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2)/(1 - q_1^2 - q_2^2)^2$, the extra terms reduce to a constant and the Hamiltonian is just the Weyl-ordered one

$$H = \tfrac{1}{2}(1 - q_1^2 - q_2^2)\boldsymbol{p}^2(1 - q_1^2 - q_2^2) = -\frac{\hbar^2}{2}(1 - q_1^2 - q_2^2)^2\left(\frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2}\right) \tag{2.9}$$

with

$$p_i = -\mathrm{i}\hbar(1 - q_1^2 - q_2^2)\frac{\partial}{\partial q_l}(1 - q_1^2 - q_2^2)^{-1}. \tag{2.10}$$

In a similar fashion one can write down the Hamiltonian formalism and proceed with the quantisation in the case of a Kähler manifold as well. Dynamics in any case is governed by the Hamiltonian in terms of the propagator

$$G(q'', q'; T) = \langle q''|\exp(-\mathrm{i}TH/\hbar)|q'\rangle. \tag{2.11}$$

3. Classical motion in two-dimensional spaces of constant negative curvature

There are various models which are known to be equivalent descriptions of a two-dimensional space of constant negative curvature [3]. One can go from one to the other in terms of suitable coordinate transformations. An infinite two-dimensional surface of constant negative curvature can be thought as one sheet of a two-sheeted hyperboloid embedded in Minkowski space. A natural parametrisation of such a surface (a pseudosphere) can be given in terms of pseudospherical coordinates $0<\theta<\infty$ and $0\leqslant\phi<2\pi$. The line element on the pseudosphere is $ds^2=(d\theta)^2+\sinh^2\theta(d\phi)^2$ while the invariant volume element is $d\cosh\theta\, d\phi$. The Lagrangian of a free particle moving on the pseudosphere is

$$L=\tfrac{1}{2}(\dot{\theta}^2+\sinh^2\theta\dot{\phi}^2). \tag{3.1}$$

Another description of the same space can be obtained by introducing $K=r\exp(i\phi)=\tanh(\theta/2)\exp(i\phi)$. In this parametrisation (the Poincaré disc) the Kähler property of the manifold is explicit and the line element is

$$ds^2=4\,dK\,d\bar{K}(1-K\bar{K})^{-2}=4[(dx)^2+(dy)^2](1-r^2)^{-2}.$$

The geodesic distance of two points on the Poincaré disc is given by

$$D[K,K']=\cosh^{-1}[1+2|K-K'|^2(1-K\bar{K})^{-1}(1-K'\bar{K}')^{-1}]. \tag{3.2}$$

The corresponding formula in pseudospherical coordinates is

$$D[\theta,\phi,\theta',\phi']=\cosh^{-1}[\cosh\theta\cosh\theta'-\sinh\theta\sinh\theta'\cos(\phi-\phi')]. \tag{3.3}$$

The continuous symmetries of the Poincaré disc can be realised by

$$K'=(\alpha K+\beta)/(\beta^*K+\alpha^*) \qquad |\alpha|^2-|\beta|^2=1$$

which correspond to the pseudo-unitary group SU(1, 1). Another model describing the same space is the Poincaré complex half-plane, which is obtained through the change of variables

$$\zeta=(1-K)/(1+K) \qquad K=(1-\zeta)/(1+\zeta).$$

The line element is $ds^2=4\,d\zeta\,d\bar{\zeta}(\zeta+\bar{\zeta})^{-2}=[(dx)^2+(dy)^2]x^{-2}$ $(-\infty<y<\infty, 0<x<\infty)$. The metric is again explicitly conformal.

The geodesic distance is

$$D(\zeta,\zeta')=\cosh^{-1}[1+2|\zeta-\zeta'|^2(\zeta+\bar{\zeta})^{-1}(\zeta'+\bar{\zeta}')^{-1}]. \tag{3.4}$$

The symmetries are realised by the transformation

$$\zeta'=(\alpha\zeta-i\beta)/(i\gamma\zeta+\delta) \qquad \alpha\delta-\beta\gamma=1$$

which again corresponds to SU(1, 1) or PSL(2, R).

The classical equations of motion in pseudospherical coordinates are

$$\ddot{\theta}-\tfrac{1}{2}\sinh(2\theta)\dot{\phi}=0 \tag{3.5}$$

$$\dot{\phi}\sinh^2\theta=L \tag{3.6}$$

where L is a constant of the motion. In addition, the energy is $E=\frac{1}{2}(\dot{\theta}^2+\sinh^2\theta\dot{\phi}^2)$. The solutions of the equations of motion can be expressed as $\theta=\theta(\theta(0),\phi(0),\dot{\theta}(0),\dot{\phi}(0);t)$, $\phi=\phi(\theta(0),\phi(0),\dot{\theta}(0),\dot{\phi}(0);t)$ or in terms of the constants E and L as $\theta=\theta(\theta(0),\phi(0),E,L;t)$ $\phi=\phi(\theta(0),\phi(0),E,L;t)$.

They are explicitly

$$\cosh\theta(t) = a\cosh(t\sqrt{2E}+b) \tag{3.7}$$

$$\phi(t) = \phi(0) + \tan^{-1}\left(\frac{\exp(2b+2t\sqrt{2E}) + (L^2-2E)/(L^2+2E)}{2L\sqrt{2E}/(L^2+2\sqrt{E})}\right)$$

$$-\tan^{-1}\left(\frac{\exp(2b) + (L^2-2E)/(L^2+2E)}{2L\sqrt{2E}/(L^2+2\sqrt{E})}\right)$$

where

$$a \equiv (1+L^2/2E)^{1/2} \qquad b = \cosh^{-1}\left(\frac{\cosh\theta(0)}{a}\right).$$

The form of the solutions in terms of Poincaré plane coordinates was given in [2]. Although $\theta(t)$ and $\phi(t)$ appear to describe a complicated motion, we are dealing with a free particle, and since the geodesic distance is

$$D[\theta(t), \phi(t), \theta(0), \phi(0)] = t\sqrt{2E}$$

the corresponding equation for the geodesic distance will have the trivial form

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} D[\theta(t), \phi(t), \theta(0), \phi(0)] = 0.$$

The phase space of the classical motion can be parametrised in various ways using suitable variables, which exhibit different geometrical aspects of the motion. In general, a classical trajectory could be fully parametrised by three quantities, for example $\theta(0)$, $\dot\theta(0)$, $\dot\phi(0)$, ($\phi(0)$ plays a trivial role). Equivalently, we could use the values of the conserved quantities of each model. In the Poincaré disc, three conserved quantities that represent the symmetry algebra are, in cartesian Poincaré-disc coordinates,

$$B_1 = \tfrac{1}{2}(1-x^2-y^2)My + Mx \qquad B_2 = -\tfrac{1}{2}(L-x^2y^2)Mx + My$$

$$M = xp_y - yp_x \qquad p_x = \dot x/(1-x^2-y^2)^2 \qquad \text{etc.}$$

These three constants satisfy the SU(1, 1) algebra

$$\{B_1, B_2\} = -M \qquad \{M_1, B_1\} = B_2 \qquad \{M, B_2\} = -B_1.$$

The Hamiltonian, being a Casimir of SU(1, 1), is

$$H = \tfrac{1}{2}(B_1^2 + B_2^2 - M^2).$$

To have a description of the trajectories, we can express B_1, B_2 and M in terms of two angles ϕ_+, ϕ_- and the energy E as

$$B_1 = 2\sqrt{E}\cos\tfrac{1}{2}(\phi_+ + \phi_-)/\sin\tfrac{1}{2}(\phi_+ - \phi_-)$$

$$B_2 = 2\sqrt{E}\sin\tfrac{1}{2}(\phi_+ + \phi_-)/\sin\tfrac{1}{2}(\phi_+ - \phi_-)$$

$$M = 2\sqrt{E}\cot\tfrac{1}{2}(\phi_+ - \phi_-).$$

The angle ϕ_- gives the angular location of a particle which, starting from the boundary of the Poincaré disc and moving on a geodesic with constant energy E, reaches again the boundary with an angular location ϕ_+. If we go to half-plane coordinates and put the point ϕ_+ at infinity, we find exponentially fast ($\exp(-t\sqrt{2E})$) converging trajectories, while a slightest variation $\delta\phi_+$ results in exponentially ($\exp(t\sqrt{2E})$) separating trajectories as $t \to \infty$. As is usually said, the model exhibits hyperbolic flow, meaning roughly that the phase-space trajectories diverge.

4. Quantum mechanics in two-dimensional spaces of constant negative curvature

The classical Hamiltonian describing a particle on the Poincaré disc is

$$H=\frac{1}{2m}(1-K\bar{K})^2\bar{\pi}\pi \tag{4.1}$$

in terms of the canonical momenta $\pi=\dot{K}/(1-K\bar{K})^2$, $\bar{\pi}=\dot{\bar{K}}/(1-K\bar{K})^2$. The quantum Hamiltonian is given by the operator

$$H=\frac{1}{2m}(1-K\bar{K})\pi\bar{\pi}(1-K\bar{K})=-\frac{\hbar^2}{2mR^2}(1-K\bar{K})^2\frac{\partial^2}{\partial K\partial\bar{K}} \tag{4.2}$$

which is equal to the Weyl-ordered one up to a constant. The momenta are represented by

$$\begin{aligned}\pi&=-\frac{\mathrm{i}\hbar}{R}(1-K\bar{K})\frac{\partial}{\partial K}(1-K\bar{K})^{-1}\\ \bar{\pi}&=-\frac{\mathrm{i}\hbar}{R}(1-K\bar{K})\frac{\partial}{\partial\bar{K}}(1-K\bar{K})^{-1}.\end{aligned} \tag{4.3}$$

On the other hand in the $\{\theta,\phi\}$ representation we have

$$H=-\frac{\hbar^2}{2mR^2}\left(\frac{1}{\sinh\theta}\frac{\partial}{\partial\theta}\sinh\theta\frac{\partial}{\partial\theta}+\frac{1}{\sinh^2\theta}\frac{\partial^2}{\partial\phi^2}\right). \tag{4.4}$$

Here R is the radius of curvature of our space, so that all variables are dimensionless.

The Hilbert space is defined with an inner product

$$\langle\psi|\chi\rangle=\int\mathrm{d}K\,\mathrm{d}\bar{K}(1-K\bar{K})^2\psi^*(K,\bar{K})\chi(K,\bar{K})=\int_1^\infty\mathrm{d}\cosh\theta\int_0^{2\pi}\mathrm{d}\phi\,\psi^*(\theta,\phi)\chi(\theta,\phi). \tag{4.5}$$

The energy spectrum can be obtained from the solution of the eigenvalue problem

$$\left(\frac{1}{\sinh\theta}\frac{\partial}{\partial\theta}\sinh\theta\frac{\partial}{\partial\theta}+\frac{1}{\sinh^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\psi_\varepsilon(\theta,\phi)=-\frac{2mR^2}{\hbar^2}\varepsilon_\lambda\psi_\varepsilon(\theta,\phi). \tag{4.6}$$

This equation is separable and has delta-function normalisable solutions [4]

$$\psi_\lambda^m(\theta,\phi)=N_\lambda^m\exp(\mathrm{i}m\phi)P^m_{\mathrm{i}\lambda-1/2}(\cosh\theta) \tag{4.7}$$

where $m=0,\pm1,\ldots,\pm\infty$, $0\leqslant\lambda\leqslant\infty$ and $\varepsilon_\lambda=(\lambda^2+\frac{1}{4})\hbar^2/2mR^2$. The functions $P^m_{\mathrm{i}\lambda-1/2}$ are the conical functions [5]. The energy eigenvalues form a continuum starting from $\varepsilon_0=\hbar^2/8mR^2$ and extending to infinity. The eigenfunctions ψ_λ^m, describing the free particle on the entire pseudosphere, form a complete orthonormal set

$$\sum_{m=-\infty}^{\infty}\int_0^\infty\mathrm{d}\lambda\,\psi_\lambda^m(\theta,\phi)\psi_\lambda^{m*}(\theta',\phi')=\delta(\phi-\phi')\delta(\cosh\theta-\cosh\theta') \tag{4.8}$$

$$\int_1^\infty\mathrm{d}\cosh\theta\int_0^{2\pi}\mathrm{d}\phi\,\psi_\lambda^{m*}(\theta,\phi)\psi_{\lambda'}^{m'}(\theta,\phi)=\delta_{mm'}\delta(\lambda-\lambda') \tag{4.9}$$

provided

$$N_\lambda^m=\left(\frac{2\pi}{\lambda\tanh(\pi\lambda)}\right)^{1/2}\frac{\Gamma(\mathrm{i}\lambda+\frac{1}{2})}{\Gamma(\mathrm{i}\lambda+m+\frac{1}{2})}. \tag{4.10}$$

The time evolution will be described by the propagator

$$G(\cosh\theta, \phi, \cosh\theta', \phi'; t)$$
$$= \sum_{m=-\infty}^{\infty} \int_0^\infty d\lambda \frac{2\pi}{\lambda\tanh(\pi\lambda)} \left|\frac{\Gamma(i\lambda+\frac{1}{2})}{\Gamma(i\lambda+m+\frac{1}{2})}\right|^2 \exp[im(\phi-\phi')] P^m_{i\lambda-1/2}(\cosh\theta)$$
$$\times P^m_{i\lambda-1/2}(\cosh\theta') \exp[-i(\lambda^2+\tfrac{1}{4})\hbar t/2mR^2]. \tag{4.11}$$

The summation over the angular momenta m can be performed (appendix 2) and we are led to

$$G(D; t) = \frac{\theta(t)}{2\pi} \int_0^\infty d\lambda\, \lambda \tanh(\pi\lambda) P_{i\lambda-1/2}(\cosh D) \exp[-i(\lambda^2+\tfrac{1}{4})\hbar t/2mR^2] \tag{4.12}$$

where

$$D = \cosh^{-1}[\cosh\theta\cosh\theta' - \sinh\theta\sinh\theta'\cos(\phi-\phi')] \tag{4.13}$$

is the geodesic distance between the initial and final point on the pseudosphere.

The propagator solves the Schrödinger equation

$$\left(H - i\frac{\partial}{\partial t}\right) G(D; t)$$
$$= \left(-\frac{\hbar^2}{2mR^2}\frac{1}{\sinh D}\frac{\partial}{\partial D}\sinh D\frac{\partial}{\partial D} - i\hbar\frac{\partial}{\partial t}\right) G(D; t)$$
$$= \frac{1}{2\pi}\int_0^\infty d\lambda\, \lambda\tanh(\pi\lambda) P_{i\lambda-1/2}(\cosh D)$$
$$\times \left\{-i\hbar\delta(t) - \left[i\hbar\theta(t)\left(-\frac{i\hbar}{2mR^2}\right)(\lambda^2+\tfrac{1}{4})\right.\right.$$
$$\left.\left. - \theta(t)\frac{\hbar^2}{2mR^2}(\lambda^2+\tfrac{1}{4})\right] \exp[-i(\lambda^2+\tfrac{1}{4})\hbar t/2mR^2]\right\}$$
$$= -i\hbar\delta(t)\delta(\cosh\theta - \cosh\theta')\delta(\phi-\phi'). \tag{4.14}$$

Here we used the property

$$\left(\frac{1}{\sinh\theta}\frac{\partial}{\partial\theta}\sinh\theta\frac{\partial}{\partial\theta} + \sinh^{-2}\theta\frac{\partial^2}{\partial\phi^2}\right) f(D(\theta,\phi,\theta',\phi')) = \frac{1}{\sinh D}\frac{2}{\partial D}\left(\sinh D\frac{\partial f}{\partial D}\right)$$

and [5] for the representation of the delta function in terms of conical functions.

The Fourier transform of $G(D; t)$ takes the form

$$\tilde{G}(D; \omega) = \int_{-\infty}^\infty dt\, \exp(i\omega t) G(D; t)$$
$$= \frac{1}{2\pi}\int_0^\infty d\lambda\, \lambda\tanh(\pi\lambda) P_{i\lambda-1/2}(\cosh D) \int_0^\infty dt\, \exp[i\omega - i(\lambda^2+\tfrac{1}{4})\hbar t/2mR^2]$$
$$= \frac{i}{2\pi}\int_0^\infty d\lambda\, \lambda\tanh(\pi\lambda) \frac{P_{i\lambda-1/2}(\cosh D)}{\omega - (\hbar/2mR^2)(\lambda^2+\frac{1}{4}) + i\varepsilon}. \tag{4.15}$$

If we use the relation [3]

$$-i\pi\tanh(\pi\lambda) P_{i\lambda-1/2} = Q_{i\lambda-1/2} - Q_{-i\lambda-1/2}$$

where Q is the conical function of the second kind, which although a solution of the time-independent Schrödinger equation, is not an eigenfunction since it is not normalisable on the entire pseudosphere, (4.15) becomes

$$\tilde{G}(D;\omega)=\frac{mR^2}{4\pi^2\hbar}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\lambda\,\frac{Q_{\mathrm{i}\lambda-1/2}(\cosh D)-Q_{-\mathrm{i}\lambda-1/2}(\cosh D)}{\lambda^2-[f(\omega)+\mathrm{i}\varepsilon]}$$
$$=\frac{mR^2}{8\pi^2\hbar}\int_{-\infty}^{\infty}\mathrm{d}x\left(\frac{Q_{\mathrm{i}\sqrt{x}-1/2}(\cosh D)}{x-f(\omega)-\mathrm{i}\varepsilon}-\frac{Q_{-\mathrm{i}\sqrt{x}-1/2}(\cosh D)}{x-f(\omega)-\mathrm{i}\varepsilon}\right)$$

where

$$f(\omega)=2mR^2(\omega-\hbar/8mR^2)/\hbar.$$

Using the asymptotic forms

$$Q_{\mathrm{i}\sqrt{x}-1/2}(\cosh D)\underset{x\to\infty}{\sim}x^{-1/4}\exp(-\mathrm{i}D\sqrt{x})$$

$$Q_{-\mathrm{i}\sqrt{x}-1/2}(\cosh D)=[Q_{\mathrm{i}\sqrt{x}-1/2}(\cosh D)]^*\underset{x\to\infty}{\sim}x^{-1/4}\exp(\mathrm{i}D\sqrt{x})$$

and going to the complex x plane, we find that the first integral is zero (by closing the contour in the lower half-plane) whereas the second integral (by closing the contour in the upper half-plane) gives

$$\tilde{G}(D;\omega)=\frac{mR^2}{4\mathrm{i}\pi\hbar}Q_{-\mathrm{i}\sqrt{f(\omega)}-1/2}(\cosh D). \tag{4.16}$$

Notice that the Fourier transform $\tilde{G}$ has a branch point at $\omega_b=\hbar/8mR^2$, implying that the energy spectrum is continuous with a lower bound $E_{\min}=\hbar\omega_b=\hbar^2/8mR^2$, in agreement with the eigenvalues of the Schrödinger equation.

In terms of the propagator $G(D;t)$ the time evolution of states can be written as

$$\psi(\theta,\phi,t)=\int_1^{\infty}\mathrm{d}\cosh\theta'\int_0^{2\pi}\mathrm{d}\phi'\,G(D(\theta,\phi;\theta',\phi';t))\psi(\theta',\phi';0). \tag{4.17}$$

Equivalently, time evolution could be studied in the Heisenberg picture starting from the Heisenberg equations of motion, which, however, are rather difficult to solve.

The classical action between two configurations $q^i(T)$ and $q^i(0)$ is given for a free particle in terms of the geodesic distance

$$S_c[q^i(T),q^i(0)]=D^2[q^i(T),q^i(0)]/2T.$$

In the semiclassical approximation, the propagator is given by

$$G(q^i(T),q^i(0);T)$$
$$=-\frac{\mathrm{i}mR^2}{2\pi\hbar}\left[\det\left(\frac{\partial^2 S_c}{\delta q^i(T)\delta q^j(0)}\right)\right]^{1/2}\exp(\mathrm{i}/\hbar S_c)(1+\mathrm{O}(\hbar)).$$

In our case, we compute

$$\det\begin{vmatrix}\dfrac{\delta^2 S_c}{\delta\cosh\theta(T)\delta\cosh\theta(0)} & \dfrac{\delta^2 S_c}{\delta\cosh\theta(T)\delta\phi(0)}\\ \dfrac{\delta^2 S_c}{\delta\phi(T)\delta\cosh\theta(0)} & \dfrac{\delta^2 S_c}{\delta\phi(T)\delta\phi(0)}\end{vmatrix}=\frac{1}{T^2}\frac{D}{\sinh D}$$

which gives

$$G(D;\,T)\simeq\frac{1}{2\mathrm{i}\pi\hbar}\frac{mR^2}{T}\left(\frac{D}{\sinh D}\right)^{1/2}\exp\left(\frac{\mathrm{i}mR^2D^2}{2\hbar T}\right)(1+\mathrm{O}(\hbar)). \tag{4.18}$$

The power of the time factor in front reveals the dimensionality of the system.

In order to estimate higher orders in $\hbar$, we can substitute $G(D;\,T)$ in the Schrödinger equation (4.14) and compare powers in $\hbar$:

$$\left(-\frac{\hbar^2}{2mR^2}\frac{1}{\sinh D}\frac{\partial}{\partial D}\sinh D\frac{\partial}{\partial D}-\mathrm{i}\hbar\frac{\partial}{\partial T}\right)$$
$$\times\frac{mR^2}{2\pi\mathrm{i}\hbar}\frac{1}{T}\left(\frac{D}{\sinh D}\right)^{1/2}\exp\left(\frac{\mathrm{i}mR^2D^2}{2\hbar T}\right)[1+\mathrm{i}\hbar TF(D)/mR^2]\simeq 0$$

where $F(D)$ is an unknown function to be determined. It is important to keep in mind that the small-$\hbar$ approximation is a large-distance approximation and should not be valid at small D. The order-$\hbar$ correction can be obtained as

$$F(D)=-\frac{1}{4}\left(1-\frac{1}{D^2}+\frac{2}{D}\frac{1}{\mathrm{e}^{2D}-1}\right)$$

plus, of course, other terms of order $\hbar^2$.

The semiclassical propagator falls off at large distance exponentially like

$$\frac{1}{T}\sqrt{D}\exp(-D/2)\left(1-\frac{\mathrm{i}\hbar}{4mR^2}T\right)\exp(\mathrm{i}mR^2D^2/2\hbar T)$$

in contrast to a particle on a plane, which propagates with $(1/T)\exp(\mathrm{i}mR^2D^2/2\hbar T)$.

The semiclassical approximation to the propagator can also be obtained from the integral expression of the conical function [5]:

$$P_{\mathrm{i}\lambda-1/2}(\cosh D)=\frac{\sqrt{2}}{\pi}\coth(\pi\lambda)\int_D^\infty\mathrm{d}u\,\sin(\lambda u)(\cosh u-\cosh D)^{-1/2}.$$

Substituting this expression in the integral form of the propagator, (4.12), we get

$$\begin{aligned}G(D;\,t)&=\frac{\theta(t)}{\sqrt{2}\pi^2}\int_D^\infty\mathrm{d}u(\cosh u-\cosh D)^{-1/2}\int_0^\infty\mathrm{d}\lambda\,\lambda\,\sin(\lambda u)\exp(-\mathrm{i}\varepsilon_\lambda t/\hbar)\\&=\frac{\theta(t)}{\sqrt{2}\pi^2}\frac{\sqrt{\pi}}{4}\left(\frac{\mathrm{i}\hbar t}{2mR^2}\right)^{-3/2}\int_D^\infty\mathrm{d}u\,u(\cosh u-\cosh D)^{-1/2}\\&\quad\times\exp\left(-\frac{mR^2u^2}{2\mathrm{i}\hbar t}\right)\exp\left(-\frac{\mathrm{i}\hbar t}{8mR^2}\right).\end{aligned} \tag{4.19}$$

With the change of variables $u=\omega^2+D$, the integral becomes

$$\begin{aligned}I&=\exp(-mR^2D^2/2\mathrm{i}\hbar t)\int_{-\infty}^\infty\mathrm{d}\omega(\omega^2+D)\left(\frac{\cosh(\omega^2)-1}{\omega^2}\cosh D+\frac{\sinh(\omega^2)}{\omega^2}\sinh D\right)^{-1/2}\\&\quad\times\exp\left(-\frac{mR^2\omega^2D}{\mathrm{i}\hbar t}\right)\exp\left(-\frac{mR^2\omega^4}{2\mathrm{i}\hbar t}\right)\\&\equiv\exp(\mathrm{i}mR^2D^2/2\hbar t)\int_{-\infty}^\infty\mathrm{d}\omega\,f(\omega)\exp(-mR^2\omega^2D/\mathrm{i}\hbar t).\end{aligned}$$

Expanding $f(\omega)=f(0)+f'(0)\omega+\ldots$ and noticing that $f(0)=D/\sqrt{\sinh D}$ we obtain

$$I=\exp(\mathrm{i}mR^2D^2/2\hbar t)\frac{D}{\sqrt{\sinh D}}\left(\int_{-\infty}^{\infty}\mathrm{d}\omega\,\exp(-mR^2D\omega^2/\mathrm{i}\hbar t)+\mathrm{i}t\mathrm{O}(\hbar)\right).$$

Thus

$$G(D;t)=\frac{\theta(t)}{2\pi\mathrm{i}}\frac{mR^2}{\hbar t}\left(\frac{D}{\sinh D}\right)^{1/2}$$
$$\times\exp(-\mathrm{i}\hbar t/8mR^2)\exp(\mathrm{i}mR^2D^2/2\hbar t)(1+\mathrm{i}t\mathrm{O}(\hbar)) \tag{4.20}$$

which agrees with the result of the semiclassical formula apart from the zero-point energy term which, in any case, to this order is unity.

A test of how good an approximation this is to the exact propagator is provided by a comparison of (4.20) with (4.19), the latter being evaluated numerically to any degree of accuracy desired. Introducing the dimensionless Euclidean time

$$\tau\equiv\mathrm{i}\hbar t/2mR^2\equiv\mathrm{i}\tilde{t}$$

the exact propagator, (4.19), can be written as

$$G(D;\tau)=(2\pi)^{-3/2}$$
$$\times\exp(-\tau/4)\tau^{-3/2}\tfrac{1}{2}\int_D^{\infty}\mathrm{d}u\,u\exp(-u^2/4\tau)(\cosh u-\cosh D)^{-1/2}$$
$$=(2\pi)^{-3/2}\exp(-\tau/4)\tau^{-3/2}\int_D^{\infty}\mathrm{d}u(\cosh u-\cosh D)^{1/2}$$
$$\times\frac{\exp(-u^2/4\tau)}{\sinh u}\left(u\coth u-1+\frac{u^2}{2\tau}\right)$$

by integration by parts. The integrand of the above expression is a smooth function of u which vanishes at the lower and upper bounds and can thus be calculated numerically to any degree of accuracy.

The result of this calculation $G_{\mathrm{exact}}(D;\tau)(2\pi)^{3/2}\exp(\tau/4)\tau^{3/2}$ is shown in figure 1 (full curve) together with the semiclassical approximation (broken curve) as a function of D for different values of τ. We observe that for $\tau\leqslant 1$, the classical approximation is very good for all D. It is also very good for any D and τ such that $D/\sqrt{\tau}\gtrsim 1$. It fails progressively more and more as τ increases ($\tau\gg 1$) and $D/\sqrt{\tau}<1$. For example, for $\tau=10^2$ it fails for $D<10$ by a factor of 2-4.

A more direct way of obtaining the propagator is by making use of the Mehler-Fock transform. Consider the Schrödinger equation for $G(D,t)$ (equation (4.14)). Take the Mehler-Fock transform [3] in D and the Fourier transform in t, i.e.

$$G(D;t)=\int_0^{\infty}\mathrm{d}\lambda\,\lambda\tanh(\pi\lambda)P_{-1/2+\mathrm{i}\lambda}(\cosh D)\int_{-\infty}^{\infty}\mathrm{d}\omega\,\exp(\mathrm{i}\omega t)\tilde{G}(\lambda,\omega)$$

and

$$\delta(t)\delta(\cosh\theta-\cosh\theta')\delta(\phi-\phi')$$
$$=\frac{1}{2\pi}\int\mathrm{d}\lambda\,\lambda\tanh(\pi\lambda)P_{-1/2+\mathrm{i}\lambda}(\cosh D)\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\exp(\mathrm{i}\omega t).$$

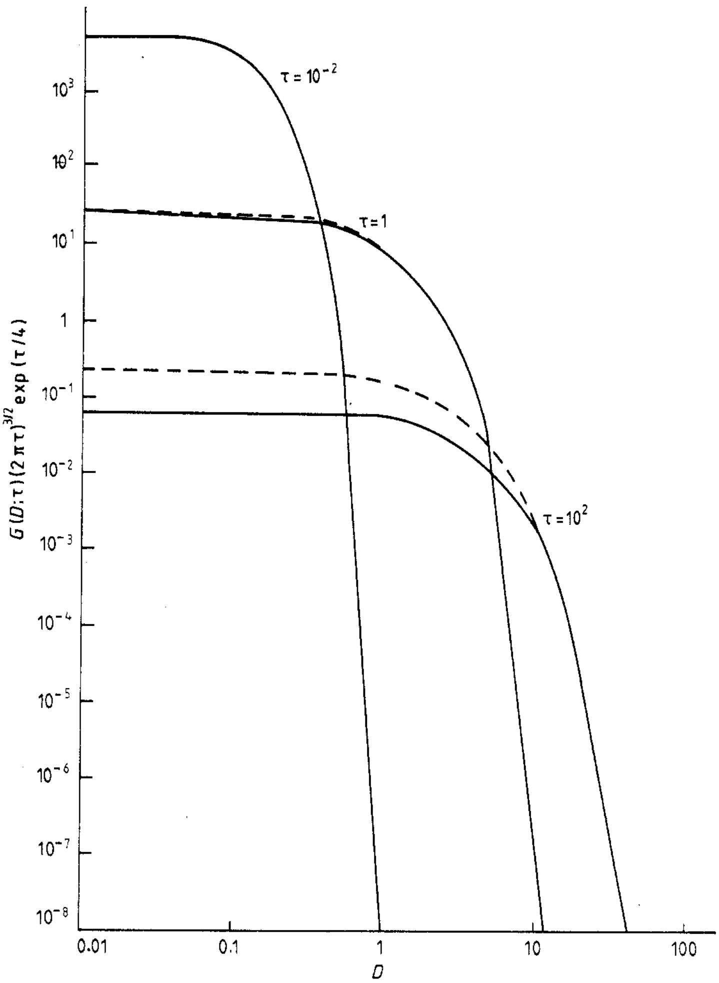

Figure 1. The exact (equation (4.19), full curve) and the semiclassical (equation (4.20), broken curve) propagators as functions of D for different values of τ; for $\tau=10^{-2}$ the two cannot be distinguished in this scale.

Then

$$(H-\mathrm{i}\hbar\partial/\partial t)G(D;\,t)$$

$$=\int_0^\infty \mathrm{d}\lambda\ \lambda\ \tanh(\pi\lambda)\int_{-\infty}^{\infty}\mathrm{d}\omega\ \exp(\mathrm{i}\omega t)\tilde{G}(\lambda,\,\omega)[H-\mathrm{i}\hbar(\mathrm{i}\omega)]P_{-1/2+\mathrm{i}\lambda}$$

$$=-\frac{\mathrm{i}\hbar}{(2\pi)^2}\int_0^\infty \mathrm{d}\lambda\ \lambda\ \tanh(\pi\lambda)\int_{-\infty}^{\infty}\mathrm{d}\omega\ \exp(\mathrm{i}\omega t)P_{-1/2+\mathrm{i}\lambda}.$$

Using the fact that [2]

$$HP_{-1/2+\mathrm{i}\lambda}=\varepsilon_\lambda P_{-1/2+\mathrm{i}\lambda}$$

where $\varepsilon_\lambda=(\lambda^2+\frac{1}{4})\hbar^2/2mR^2$, we obtain

$$\tilde{G}(\lambda,\,\omega)=-\frac{\mathrm{i}\hbar}{(2\pi)^2}\frac{1}{\hbar\omega+\varepsilon_\lambda-\mathrm{i}\varepsilon}.$$

The iε prescription follows from the fact that

$$-\frac{\mathrm{i}\hbar}{2\pi^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega\,\exp(\mathrm{i}\omega t)}{\hbar\omega+\varepsilon_\lambda-\mathrm{i}\varepsilon}=\frac{1}{2\pi}\exp(-\mathrm{i}\varepsilon_\lambda t/\hbar)\theta(t)$$

so that G obeys the causality condition. We thus obtain

$$G(D;\,t)=\frac{\theta(t)}{2\pi}\int_0^\infty \mathrm{d}\lambda\,\lambda\,\tanh(\pi\lambda)P_{\mathrm{i}\lambda-1/2}(\cosh D)\exp(-\mathrm{i}\varepsilon_\lambda t/\hbar)$$

which is (4.12), since $\varepsilon_\lambda/\hbar=(\lambda^2+\frac{1}{4})\hbar/2mR^2$. If we now use the addition formula for the P (appendix 2) in (4.12) and compare the resulting expression with (4.11) for G, we conclude immediately that the normalised energy eigenfunctions are those given by (4.7).

We carry out next the quantisation in the Poincaré half-plane in order to compare the results with those obtained on the pseudosphere. In half-plane variables the classical Lagrangian is

$$L_c=\tfrac{1}{2}mR^2(\dot{x}^2+\dot{y}^2)/x^2 \tag{4.21}$$

leading to the classical Hamiltonian

$$H_c=(1/2mR^2)x^2(p_x^2+p_y^2). \tag{4.22}$$

Using

$$g_{ij}=\begin{vmatrix}\dfrac{1}{x^2} & 0\\ 0 & \dfrac{1}{x^2}\end{vmatrix}$$

we get the following representation of the momentum operators:

$$p_x=-\mathrm{i}\hbar\left(\frac{\partial}{\partial x}-\frac{1}{x}\right)\qquad p_y=-\mathrm{i}\hbar\frac{\partial}{\partial y}. \tag{4.23}$$

So that the time-independent Schrödinger equation becomes

$$-\frac{\hbar^2}{2mR^2}x^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi(x,y)=E\psi(x,y). \tag{4.24}$$

Letting

$$\psi(x,y)=\exp(\mathrm{i}ky)x^{1/2}g(x)$$

we obtain an equation for $g(x)$:

$$x^2g''+xg'-[k^2x^2-(2mR^2/\hbar^2)E+\tfrac{1}{4}]g=0.$$

Letting

$$p=\left[\frac{2mR^2}{\hbar^2}\left(E-\frac{\hbar^2}{8mR^2}\right)\right]$$

the equation becomes

$$x^2g''+xg'-[k^2x^2+(\mathrm{i}p)^2]g=0. \tag{4.25}$$

This is the modified Bessel equation, which has solutions for any real p, i.e. $E \geqslant \hbar^2/8mR^2$. The normalised solution is

$$g(x) = \left(\frac{1}{\pi^3} p \sinh(\pi p)\right)^{1/2} K_{\mathrm{i}p}(|k|x). \tag{4.26}$$

Since $x^{1/2}K_{\mathrm{i}p}(x) \to_{x\to\infty} 0$, the value $k=0$ is excluded.

We see then that in the Poincaré half-plane quantisation gives the same energy spectrum, but both quantum numbers are now continuous, in contrast to the pseudosphere case where m is integer. The same results have been obtained by the path integral method [6].

5. Expectation values and time evolution

Wavefunctions describing the system under study can be expressed as a superposition of energy eigenfunctions

$$\begin{aligned}\psi(\theta,\phi) &= \sum_{m=-\infty}^{\infty} \int \mathrm{d}\lambda\, C_m(\lambda)\left(\frac{1}{2\pi}\lambda \tanh(\pi\lambda)\right)^{-1/2} \exp(\mathrm{i}m\phi) B^m_{\mathrm{i}\lambda-1/2}(\cosh\theta) \\ &= \sum_{m=-\infty} \int_0^\infty \mathrm{d}\lambda\, C_m(\lambda)\left(\frac{\lambda\tanh(\pi\lambda)}{2\pi}\right)^{-1/2} \frac{\Gamma(\mathrm{i}\lambda+\frac{1}{2})}{\Gamma(\mathrm{i}\lambda+m+\frac{1}{2})} \\ &\quad\times \exp(\mathrm{i}m\phi) P^m_{\mathrm{i}\lambda-1/2}(\cosh\theta). \end{aligned} \tag{5.1}$$

The coefficients $C_m(\lambda)$ are expressible in terms of $\psi(\theta,\phi)$ as

$$C_m(\lambda) = \left(\frac{\lambda\tanh(\pi\lambda)}{2\pi}\right) \int_1^\infty \mathrm{d}\cosh\theta \int_0^{2\pi} \mathrm{d}\phi\, \exp(-\mathrm{i}m\phi) B^{*m0}_{\mathrm{i}\lambda-1/2}(\cosh\theta)\psi(\theta,\phi).$$

In the case of a ϕ-independent wavefunction, the coefficients can have only $m=0$. Setting $C_m(\lambda) = \delta_{m0}C(\lambda)$, a general ϕ independent wavefunction can be written as a Mehler–Fock transformation

$$\psi(\cosh\theta) = \int_0^\infty \mathrm{d}\lambda \left(\frac{\lambda\tanh(\pi\lambda)}{2\pi}\right)^{-1/2} c(\lambda) P_{\mathrm{i}\lambda-1/2}(\cosh\theta). \tag{5.2}$$

It is also possible to consider wavefunctions depending on the geodesic distance from a fixed centre θ_0, ϕ_0. A specific choice of coefficients

$$C_m(\lambda) = C(\lambda)(-1)^m \frac{\Gamma(\mathrm{i}\lambda+m+\frac{1}{2})}{\Gamma(\mathrm{i}\lambda+\frac{1}{2})} \exp(-\mathrm{i}m\phi_0) P^{-m}_{\mathrm{i}\lambda-1/2}(\cosh\theta_0)$$

with the help of the Mehler addition formula (appendix 2) gives

$$\psi(\cosh D) = \int_0^\infty \mathrm{d}\lambda\, C(\lambda)\left(\frac{\lambda\tanh(\pi\lambda)}{2\pi}\right)^{-1/2} P_{\mathrm{i}\lambda-1/2}(\cosh D) \tag{5.3}$$

with

$$\cosh D = \cosh\theta\cosh\theta_0 - \sinh\theta_0\cos(\phi-\phi_0).$$

A family of localised wavefunctions is given by Gaussian functions of the geodesic distance from a fixed centre. For example, let us consider

$$\psi(D;0) = \left(\frac{D}{\pi\alpha\sinh D}\right)^{1/2} \exp(-(D^2-\theta_0^2)/2\alpha). \tag{5.4}$$

The time-evolved wavefunction will be given by

$$\psi(D(\theta,\phi;\theta_0,\phi_0);t)$$
$$=\int_1^\infty \mathrm{d}\cosh\theta' \int_0^{2\pi} \mathrm{d}\phi'\, G(D(\theta,\phi;\theta',\phi');t)\psi(D(\theta',\phi';\theta_0,\phi_0);0) \quad (5.5)$$

It is easy to prove that

$$G(D_{12};\tau_1-\tau_2)$$
$$=\int_0^\infty \mathrm{d}\cosh\theta_3 \int_0^{2\pi} \mathrm{d}\phi_3\, G(D_{13};\tau_1-\tau_3)G(D_{32};\tau_3-\tau_2). \quad (5.6)$$

Using the semiclassical expression, (4.18), for the propagator in the above equation, we get

$$\frac{1}{4\pi(\tau_1-\tau_2)}\left(\frac{D_{12}}{\sinh D_{12}}\right)^{1/2} \exp[-D_{12}^2/4(\tau_1-\tau_2)]$$
$$\simeq\frac{1}{(4\pi)^2}\frac{1}{(\tau_1-\tau_3)(\tau_3-\tau_2)}\int_1^\infty \mathrm{d}\cosh\theta_3\int_0^{2\pi}\mathrm{d}\phi_3$$
$$\times\exp[-D_{13}^2/4(\tau_1-\tau_3)]\exp[-D_{32}^2/4(\tau_3-\tau_2)]\left(\frac{D_{13}D_{32}}{\sinh D_{13}\sinh D_{32}}\right)^{1/2}.$$

For $\tau_3=0$ and $\tau_2=-\alpha/2$ this gives

$$\frac{1}{\tau_1+(\alpha/2)}\left(\frac{D_{12}}{\sinh D_{12}}\right)^{1/2}\exp[-D_{12}^2/4(\tau_1+\tfrac{1}{2}\alpha)]$$
$$\simeq\frac{2\sqrt{\pi}}{\sqrt{\alpha}}\int_1^\infty \mathrm{d}\cosh\theta_3\int_0^{2\pi}\mathrm{d}\phi_3\left[\frac{1}{4\pi\tau_1}\exp(-D_{13}^2/4\tau_1)\left(\frac{D_{13}}{\sinh D_{13}}\right)^{1/2}\right]$$
$$\times\exp(-\theta_0^2/2\alpha)\left[\frac{1}{\sqrt{\pi\alpha}}\left(\frac{D_{32}}{\sinh D_{32}}\right)^{1/2}\exp(-D_{32}^2/2\alpha)\exp(\theta_0^2/2\alpha)\right]$$
$$=2\sqrt{\frac{\pi}{\alpha}}\exp(-\theta_0^2/2\alpha)\int_1^\infty \mathrm{d}\cosh\theta_3\int_0^{2\pi}\mathrm{d}\phi_3\, G(D_{13};\tau_1)\psi(D_{32};0)$$

which means that (see equation (5.5))

$$\psi(D;\tau)=\psi(\theta,\phi,\theta_0,\phi_0;t)$$
$$=\frac{\alpha}{\alpha+2\tau}\frac{1}{\sqrt{\pi\alpha}}\left(\frac{D}{\sinh D}\right)^{1/2}\exp(\theta_0^2/2\alpha)\exp[-D^2/2\alpha(1+2\tau/\alpha)]$$
$$=\frac{\alpha}{\alpha+\mathrm{i}\hbar t/mR^2}\frac{1}{\sqrt{\pi\alpha}}\left(\frac{D}{\sinh D}\right)^{1/2}\exp(\theta_0^2/2\alpha)\exp\left[-D^2/2\alpha\left(1+\frac{\mathrm{i}\hbar t}{\alpha mR^2}\right)\right].$$

Since $\langle\psi(D;\tilde{t})|\psi(D;\tilde{t})\rangle=\exp(\theta_0^2\tilde{t}^2/\alpha(1+\tilde{t}^2))$, normalisation is not quite preserved for $\tau\gg1$ in this approximation, unless θ_0 is very small. Expectation values of

D-dependent observables are readily computed. For instance, we find

$$\begin{aligned}\langle\psi(D;\,\tilde t)|D|\psi(D;\,\tilde t)\rangle &= \theta_0\exp(\theta_0^2/\alpha)\exp[-\theta_0^2\alpha/(\alpha^2+\tilde t^2)]\\ &\quad+\frac{\sqrt{\pi}}{2}\exp(\theta_0^2/\alpha)\left(\frac{\alpha^2+\tilde t^2}{\alpha}\right)^{1/2}\operatorname{erfc}\left(\theta_0\sqrt{\frac{\alpha}{\alpha^2+\tilde t^2}}\right)\end{aligned}$$

$$\langle\psi(D;\,\tilde t)/D^2/\psi(D;\,\tilde t)\rangle=\exp(\theta_0^2/\alpha)\exp[-\theta_0^2\alpha/(\alpha^2+\tilde t^2)]\left(\theta_0^2+\frac{\alpha^2+\tilde t^2}{\alpha}\right).$$

For $t\gg 1$ we have

$$\frac{\langle\psi(D;\,\tilde t)|D|\psi(D;\,\tilde t)\rangle}{\langle\psi(D;\,\tilde t)|\psi(D;\,\tilde t)\rangle}\equiv\langle D\rangle_{\tilde t}=\frac{\sqrt{\pi}}{2\sqrt{\alpha}}\,\tilde t\equiv\frac{\sqrt{\pi}}{\sqrt{\alpha}}\,\frac{\hbar t}{4mR^2}$$

$$\langle D^2\rangle_{\tilde t}\simeq\frac{1}{\alpha}\,\tilde t^2\equiv\frac{1}{\alpha}\,\frac{\hbar^2t^2}{4m^2R^4}$$

$$(\Delta D)^2=\frac{1}{\alpha}\left(1-\frac{\pi}{4}\right)\tilde t^2\equiv\frac{1}{\alpha}\left(1-\frac{\pi}{4}\right)\frac{\hbar^2t^2}{4m^2R^4}.$$

We see that $\langle D\rangle_{\tilde t}$ grows linearly with time just as in the classical problem where $D(t)=t\sqrt{2E}$.

It is interesting to also study the time evolution of the position operators θ and ϕ or the Poincaré-plane ones

$$\zeta_{\mathrm R}=(\zeta+\bar\zeta)/2=(\cosh\theta-\sinh\theta\cos\phi)^{-1}$$

$$\zeta_{\mathrm I}=(\zeta-\bar\zeta)/2\mathrm{i}=-\sin\phi(\coth\theta+\cos\phi)^{-1}.$$

Considering a Gaussian state, properly normalised for $\tilde t\gg 1$,

$$\psi(D;\,\tilde t)=\frac{\alpha}{\alpha+\mathrm{i}\tilde t}\,\frac{1}{\sqrt{\alpha\pi}}\left(\frac{D}{\sinh D}\right)^{1/2}\exp[-D^2/2(\alpha+\mathrm{i}\tilde t)]$$

we get

$$\begin{aligned}\langle\zeta_{\mathrm R}\rangle_{\tilde t}&=\frac{\alpha}{\pi}\,\frac{1}{\alpha^2+\tilde t^2}\int_0^\infty \mathrm d\cosh\theta\int_0^{2\pi}\mathrm d\phi\,\frac{D}{\sinh D}\\ &\qquad\times\exp[-\alpha D^2/(\alpha^2+\tilde t^2)](\cosh\theta-\sinh\theta\cos\phi)^{-1}\end{aligned}$$

$$\begin{aligned}\langle\zeta_{\mathrm I}\rangle_{\tilde t}&=-\frac{\alpha}{\pi}\,\frac{1}{\alpha^2+\tilde t^2}\int_0^\infty \mathrm d\cosh\theta\int_0^{2\pi}\mathrm d\phi\,\frac{D}{\sinh D}\\ &\qquad\times\exp[-\alpha D^2/(\alpha^2+\tilde t^2)]\sin\phi(\coth\theta+\cos\phi)^{-1}.\end{aligned}$$

The simplest special case one can think of is the case that the wavefunction is centred at $\theta_0=\phi_0=0$. In that case the ϕ dependence disappears and we get

$$\begin{aligned}\langle\zeta_{\mathrm R}\rangle_{\tilde t}&=\frac{\alpha/\pi}{\alpha^2+\tilde t^2}\int_0^\infty\mathrm d\theta\,\theta\int_0^{2\pi}\mathrm d\phi\,\exp[-\alpha\theta^2/(\alpha^2+\tilde t^2)](\cosh\theta-\sinh\theta\cos\phi)^{-1}\\ &=\frac{2\alpha}{\alpha^2+\tilde t^2}\int_0^\infty\mathrm d\theta\,\theta\exp[-\alpha\theta^2/(\alpha^2+\tilde t^2)]=1\qquad\tilde t\gg 1\end{aligned}$$

and

$$\langle\zeta_{\mathrm I}\rangle_{\tilde t}=\frac{\alpha/\pi}{\alpha^2+\tilde t^2}\int_0^\infty\mathrm d\theta\,\theta\exp[-\alpha\theta^2/(\alpha^2+\tilde t^2)]\int_{\phi=0}^{\phi=2\pi}\mathrm d\cos\phi(\coth\theta+\cos\phi)^{-1}=0.$$

It can actually be proven that $\langle\zeta_R\rangle_{\tilde{t}}=1$ and $\langle\zeta_I\rangle_{\tilde{t}}=0$ for any ϕ-independent wavefunction.

In order to see what happens for ϕ-dependent wavefunctions let us consider the limiting case of Gaussian wavefunctions centred at infinity (i.e. $\phi_0=0$, $\theta_0\to\infty$). Then

$$\cosh D \simeq \tfrac{1}{2}\exp(\theta_0)(\cosh\theta-\sinh\theta\cos\phi)$$

and

$$\langle\zeta_R\rangle_{\tilde{t}} \simeq \frac{\alpha/\pi}{\alpha^2+\tilde{t}^2}\int_1^\infty \mathrm{d}\cosh\theta\int_0^{2\pi}\mathrm{d}\phi\frac{D}{\sinh D}\exp[-\alpha D^2/(\alpha^2+\tilde{t}^2)](2\cosh D\exp(-\theta_0))^{-1}$$
$$=\frac{\alpha\exp(\theta_0)}{\alpha^2+\tilde{t}^2}\int_{\theta_0}^\infty \mathrm{d}D\frac{D}{\cosh D}\exp[-\alpha D^2/(\alpha^2+\tilde{t}^2)]$$

where we used appendix 3. Let $A=[\alpha/(\alpha^2+\tilde{t}^2)]$ and, since $\theta_0\gg 2$, we obtain

$$\langle\zeta_R\rangle_{\tilde{t}}=\exp(\theta_0)A^2\int_0^\infty \mathrm{d}D\frac{D}{\cosh D}\exp(-A^2D^2)$$
$$\simeq 2\exp(\theta_0)A^2\int_0^\infty \mathrm{d}D\,D\exp(D)\exp(-A^2D^2)$$
$$=2\exp(\theta_0)\exp(1/4A^2)\int_{A\theta_0+1/2A}^\infty \mathrm{d}x(x-\tfrac{1}{2}A)\exp(-x^2)$$
$$=\exp(\theta_0)\exp(1/4A^2)\left\{\exp\left[-\left(A\theta_0+\frac{1}{2A}\right)^2\right]-\frac{\sqrt{\pi}}{2A}\operatorname{erfc}\left(A\theta_0+\frac{1}{2A}\right)\right\}.$$

Assuming we can extrapolate to large times and using the asymptotic form of erfc:

$$\operatorname{erfc}(z)\underset{z\to\infty}{\to}\frac{1}{z\sqrt{\pi}}\exp(-z^2)\left[1-\frac{1}{2z^2}+\mathrm{O}\left(\frac{1}{z^4}\right)\right]$$

we see that time dependence persists, namely

$$\langle\zeta_R\rangle_{\tilde{t}}/\langle\zeta_R\rangle_0\simeq\alpha^2/(\alpha^2+\tilde{t}^2)+\mathrm{O}(\alpha(\tilde{t}^4).$$

Thus we see that in the ϕ-dependent case $\langle\zeta_R\rangle$ depends on time, going to zero as $t\to\infty$, in contrast to the case of the ϕ-independent wavefunction, where $\langle\zeta_R\rangle_{\tilde{t}}\to 1$. This situation is, of course, analogous to the classical behaviour [2]. Thus we can conclude that motion in the semiclassical approximation does not deviate qualitatively from the classical motion.

Appendix 1

As an example of a space of constant negative curvature consider the Kähler manifold with $G=-(1/K^2)\ln(1-K^2Z^A\bar{Z}_A)$ $(A=1,\ldots,n)$. The metric is

$$g_A^B\equiv\frac{\partial^2}{\partial Z^A\partial\bar{Z}_B}G=\frac{1}{1-K^2Z^A\bar{Z}_A}\left(\delta_A^B+K^2\frac{\bar{Z}_AZ^B}{1-K^2Z^A\bar{Z}_A}\right)$$

Then

$$\det g_A^B=\exp\operatorname{Tr}\ln\left[\frac{1}{1-K^2\bar{Z}\cdot Z}\left(\delta_A^B+K^2\frac{\bar{Z}_AZ^B}{1-K^2\bar{Z}\cdot Z}\right)\right]$$
$$=\exp\left[-n\ln(1-K^2\bar{Z}Z)+\operatorname{Tr}\sum_{\nu=1}^\infty\frac{1}{\nu}\frac{(\bar{Z}_AZ_B)^\nu}{(1-K^2\bar{Z}Z)^\nu}(K^2)^\nu\right]$$
$$=\exp[-n\ln(1-K^2\bar{Z}Z)-\ln(1-K^2\bar{Z}Z)].$$

Therefore

$$\det g_A^B = (1-K^2\bar{Z}Z)^{-(n+1)}.$$

Finally the curvature will be

$$R_A^B = -\frac{\partial}{\partial Z^A}\frac{\partial}{\partial \bar{Z}^B}\ln(1-K^2\bar{Z}Z)^{-(n+1)}$$

$$= K^2(n+1)\frac{\partial}{\partial Z^A}\frac{\partial}{\partial \bar{Z}_B}\left[-\frac{1}{K^2}\ln(1-K^2\bar{Z}Z)\right] = -K^2(n+1)g_A^B.$$

Appendix 2

The functions $B_l^{mn}(z)$ form the canonical basis for the irreducible representations of the group SL(2, C) and can be viewed as playing the same role for the group SU(1, 1). A convenient representation, which can serve as a definition for the functions $B_l^{mn}(z)$, is

$$B_l^{mn}(\cosh\theta) = \frac{1}{2\pi \mathrm{i}}\int_\Gamma \mathrm{d}z\left(\cosh\frac{\theta}{2} - z\sinh\frac{\theta}{2}\right)^{l+n}\left(\sinh\frac{\theta}{2} + z\cosh\frac{\theta}{2}\right)^{l-n} z^{m-l-1}$$

where Γ is the unit circle prescribed positively, m and n are integers and l can be complex (typically of the form $l = \pm \mathrm{i}\rho - \frac{1}{2}$, $\rho > 0$). The generating function of the B_l^{mn} is

$$\sum_{m=-\infty}^{\infty} B_l^{mn}(\cosh\theta)\exp(-\mathrm{i}m\phi)$$

$$= \exp(-\mathrm{i}n\phi)\left(\cosh\frac{\theta}{2} + \exp(i\phi)\sinh\frac{\theta}{2}\right)^{l+n}\left(\cosh\frac{\theta}{2} + \exp(-\phi)\sinh\frac{\theta}{2}\right)^{l-n}.$$

In the case $n = 0$, we have

$$\sum_{m=-\infty}^{\infty} B_{\pm \mathrm{i}\lambda - 1/2}^{m0}(\cosh\theta)\exp(-\mathrm{i}m\phi) = (\cosh\theta + \sinh\theta\cos\phi)^{\pm \mathrm{i}\lambda - 1/2}.$$

The following properties are useful

$$B_l^{mn}(\cosh\theta) = B_l^{-m,-n}(\cosh\theta)$$

$$B_l^{mn}(\cosh\theta) = (-1)^{m-n}B_{-l-1}^{mn}(\cosh\theta)$$

$$[B_l^{mn}(\cosh\theta)]^* = B_{l^*}^{mn}(\cosh\theta).$$

The Legendre functions $P_l^m(z)$ with $l = \mathrm{i}\lambda - \frac{1}{2}$ are called conical functions and are related to $B_{\mathrm{i}\lambda - 1/2}^{m0}$ through

$$B_{\mathrm{i}\lambda-1/2}^{m0}(\cosh\theta) = \frac{\Gamma(\mathrm{i}\lambda + \frac{1}{2})}{\Gamma(\mathrm{i}\lambda + m + \frac{1}{2})}P_{\mathrm{i}\lambda-1/2}^{m}(\cosh\theta).$$

The orthogonality relation

$$\int_1^\infty \mathrm{d}\cosh\theta\, B_{\mathrm{i}\lambda-1/2}^{*m0}(\cosh\theta)B_{\mathrm{i}\lambda'-1/2}^{m0}(\cosh\theta) = \frac{1}{(2\pi)^2}\lambda\tanh(\pi\lambda)\delta(\lambda-\lambda')$$

enables us to prove orthonormality for the energy eigenfunctions $\psi_\lambda^m(\theta, \phi)$. Indeed,

$$\int_1^\infty \mathrm{d}\cosh\theta \int_0^{2\pi} \mathrm{d}\phi\, \psi_\lambda^{*m}(\theta,\phi)\psi_{\lambda'}^{m'}(\theta,\phi)$$

$$= 2\pi N_\lambda^{*m} N_{\lambda'}^{m'} \delta_{mm'} \frac{\Gamma(-\mathrm{i}\lambda + m + \frac{1}{2})}{\Gamma(-\mathrm{i}\lambda + \frac{1}{2})} \frac{\Gamma(\mathrm{i}\lambda' + m + \frac{1}{2})}{\Gamma(\mathrm{i}\lambda' + \frac{1}{2})}$$

$$\times \int_1^\infty \mathrm{d}\cosh\theta\, B_{\mathrm{i}\lambda-1/2}^{*m0}(\cosh\theta) B_{\mathrm{i}\lambda'-1/2}^{m'0}(\cosh\theta) = \delta_{mm'}\delta(\lambda - \lambda')$$

provided

$$N_\lambda^m = \left(\frac{2\pi}{\lambda \tanh \pi\lambda}\right)^{1/2} \frac{\Gamma(\mathrm{i}\lambda + \frac{1}{2})}{\Gamma(\mathrm{i}\lambda + m + \frac{1}{2})}.$$

On the other hand, the propagator contains the sum over angular momenta

$$\sum_{m=-\infty}^{\infty} \left| \frac{\Gamma(\mathrm{i}\lambda + \frac{1}{2})}{\Gamma(\mathrm{i}\lambda + m + \frac{1}{2})} \right|^2 \exp[im(\phi - \phi')] P_{\mathrm{i}\lambda-1/2}^m(\cosh\theta) P_{\mathrm{i}\lambda-1/2}^m(\cosh\theta')$$

$$= P_{\mathrm{i}\lambda-1/2}(\cosh\theta) P_{\mathrm{i}\lambda-1/2}(\cosh\theta')$$

$$+ \sum_{m=1}^{\infty} \left[\exp[im(\phi-\phi')] \frac{\Gamma(-\mathrm{i}\lambda + \frac{1}{2})}{\Gamma(-\mathrm{i}\lambda + m + \frac{1}{2})} \frac{\Gamma(\mathrm{i}\lambda + \frac{1}{2})}{\Gamma(\mathrm{i}\lambda - m + \frac{1}{2})} \right.$$

$$\left. \times P_{\mathrm{i}\lambda-1/2}^m(\cosh\theta) P_{\mathrm{i}\lambda-1/2}^{-m}(\cosh\theta') \right]$$

$$+ \sum_{m=1}^{\infty} \left[\exp(-im(\phi-\phi')] \frac{\Gamma(\mathrm{i}\lambda + \frac{1}{2})}{\Gamma(\mathrm{i}\lambda + m + \frac{1}{2})} \frac{\Gamma(-\mathrm{i}\lambda + \frac{1}{2})}{\Gamma(-\mathrm{i}\lambda - m + \frac{1}{2})} \right.$$

$$\left. \times P_{\mathrm{i}\lambda-1/2}^m(\cosh\theta) P_{\mathrm{i}\lambda-1/2}^{-m}(\cosh\theta') \right]$$

$$= P_{\mathrm{i}\lambda-1/2}(\cosh\theta) P_{\mathrm{i}\lambda-1/2}(\cosh\theta') + \frac{1}{\pi} \sum_{m=1}^{\infty} P_{\mathrm{i}\lambda-1/2}^m(\cosh\theta) P_{\mathrm{i}\lambda-1/2}^{-m}(\cosh\theta')$$

$$\times \left[\frac{n \cosh(\pi\lambda) \exp[im(\phi-\phi')]}{\sin^{-1}(-\mathrm{i}\pi\lambda + m\pi + \pi/2)} + \frac{\exp[-im(\phi-\phi')]}{\sin^{-1}(\mathrm{i}\pi\lambda + \pi m + \pi/2)} \right]$$

$$= P_{\mathrm{i}\lambda-1/2}(\cosh\theta) P_{\mathrm{i}\lambda-1/2}(\cosh\theta')$$

$$+ 2 \sum_{m=1}^{\infty} (-1)^m \cos[m(\phi-\phi')] P_{\mathrm{i}\lambda-1/2}^m(\cosh\theta) P_{\mathrm{i}\lambda-1/2}^{-m}(\cosh\theta')$$

$$= P_{\mathrm{i}\lambda-1/2}(\cosh\bar{\theta})$$

where the last step is the Mehler addition formula with

$$\cosh\bar{\theta} = \cosh\theta \cosh\theta' - \sinh\theta \sinh\theta' \cos(\phi - \phi').$$

Finally, the following formula holds:

$$\int_0^\infty \mathrm{d}\lambda\, \lambda \tanh(\pi\lambda) P_{\mathrm{i}\lambda-1/2}(\cosh\bar{\theta}) = 2\pi\delta(\cosh\theta - \cosh\theta')\delta(\phi - \phi')$$

which is equivalent to the statement

$$G(D;0)=\delta(\cosh\theta-\cosh\theta')\delta(\phi-\phi').$$

Most of these formulae can be found in [5].

Appendix 3

A useful property is that

$$J\equiv\int_1^\infty \mathrm{d}\cosh\theta\int_0^{2\pi}\mathrm{d}\phi f(D)=2\pi\int_{\cosh\theta_0}^\infty \mathrm{d}\cosh D f(D).$$

The proof proceeds as follows:

$$J=\int_{\cosh\theta_0}^\infty \mathrm{d}\cosh D f(D)\int_0^{2\pi}\mathrm{d}\phi\frac{\partial\cosh\theta}{\partial\cosh D}.$$

The Jacobian is

$$\frac{\partial\cosh\theta}{\partial\cosh D}\equiv\frac{\partial\xi}{\partial\bar{\xi}}=\frac{\partial}{\partial\bar{\xi}}\left\{\frac{-\xi_0\bar{\xi}\pm\cos(\phi-\phi_0)\sqrt{\xi_0^2-1}[\cos^2(\phi-\phi_0)(\xi_0^2-1)+\bar{\xi}^2-\xi_0^2]^{1/2}}{(\xi_0^2-1)\cos^2(\phi-\phi_0)-\xi_0^2}\right\}$$

where

$$\bar{\xi}=\cosh D\qquad \xi=\cosh\theta\qquad \xi_0=\cosh\theta_0.$$

Differentiating, we get

$$\frac{\partial\xi}{\partial\bar{\xi}}=\frac{[-\xi_0\pm\bar{\xi}\cos(\phi-\phi_0)\sqrt{\xi_0^2-1}(\cos^2(\phi-\phi_0)(\xi_0^2-1)+\bar{\xi}^2-\xi_0^2)]^{-1/2}}{[(\xi_0^2-1)\cos^2(\phi-\phi_0)-\xi_0^2]}.$$

Performing the integration over ϕ we obtain zero for the second term and for the first we obtain

$$\int_0^{2\pi}\mathrm{d}\phi\frac{2\xi}{\partial\bar{\xi}}=-\xi_0\int_0^{2\pi}\frac{\mathrm{d}\phi}{(\xi_0^2-1)\cos^2(\phi-\phi_0)-\xi_0^2}$$

$$=-\cosh\theta_0\int_0^{2\pi}\frac{\mathrm{d}\phi}{\sinh^2\theta_0\cos^2\phi-\cosh^2\theta}$$

$$=-\frac{1}{2}\left[\int_0^{2\pi}\frac{\mathrm{d}\phi}{\sinh\theta_0\cos\phi-\cosh\theta_0}-\int_0^{2\pi}\frac{\mathrm{d}\phi}{\sinh\theta_0\cos\phi+\cosh\theta_0}\right]$$

$$=-\frac{1}{2\sinh\theta_0}\left[\int_0^{2\pi}\frac{\mathrm{d}\phi}{\cos\phi-\coth\theta_0}-\int_0^{2\pi}\frac{\mathrm{d}\phi}{\cos\phi+\coth\theta_0}\right]$$

$$=-\frac{1}{2\sinh\theta_0}\frac{1}{\coth\theta_0}\left[\int_0^{2\pi}\frac{\mathrm{d}\phi}{1-\tanh\theta_0\cos\phi}+\int_0^{2\pi}\frac{\mathrm{d}\phi}{1+\tanh\theta_0\cos\phi}\right]$$

$$=\frac{1}{2\cosh\theta_0}\frac{4\pi}{\sqrt{1-\tanh^2\theta_0}}$$

$$=2\pi.$$

Therefore

$$J=\int_{\cosh\theta_0}^{\infty} \mathrm{d}\cosh Df(D)\int_0^{2\pi}\mathrm{d}\phi\,\frac{\partial\xi}{\partial\bar{\xi}}$$

$$=2\pi\int_{\cosh\theta_0}^{\infty}\mathrm{d}\cosh Df(D).$$

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